

How Well Can Graphs Represent Wireless Interference?

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Abstract

Efficient use of a wireless network requires that transmissions be grouped into feasible sets, where feasibility means that each transmission can be successfully decoded in spite of the interference caused by simultaneous transmissions. Feasibility is most closely modeled by a signal-to-interference-plus-noise (SINR) formula, which unfortunately is conceptually complicated, being an asymmetric, cumulative, many-to-one relationship.

We re-examine how well graphs can capture wireless receptions as encoded in SINR relationships, placing them in a framework in order to understand the limits of such modelling. We seek for each wireless instance a pair of graphs that provide upper and lower bounds on the feasibility relation, while aiming to minimize the gap between the two graphs. The cost of a graph formulation is the worst gap over all instances, and the price of (graph) abstraction is the smallest cost of a graph formulation.

We propose a family of conflict graphs that is parameterized by a non-decreasing sub-linear function, and show that with a judicious choice of functions, the graphs can capture feasibility with a cost of $O(\log^* \Delta)$, where Δ is the ratio between the longest and the shortest link length. This holds on the plane and more generally in doubling metrics. We use this to give greatly improved $O(\log^* \Delta)$ -approximation for fundamental link scheduling problems with arbitrary power control.

We explore the limits of graph representations and find that our upper bound is tight: the price of graph abstraction is $\Omega(\log^* \Delta)$. We also give strong impossibility results for general metrics, and for approximations in terms of the number of links.

1 Introduction

Wireless scheduling. At the heart of any wireless network is a mechanism for managing *interference* between simultaneous transmissions. The medium access (MAC) layer manages access to the shared resource, the wireless spectrum, balancing the aim of maximizing simultaneous use with the impact of the resulting interference. We can represent a transmission as a communication *link*, a sender-receiver pair of nodes in a metric space. Wireless scheduling mechanisms assign the links to different “slots”, involving different frequencies, phases and/or time steps.

The model of communication that most closely captures actual conditions, nicknamed the *physical model*, uses a formula based on the ratio of the (intended) signal strength to the received interference strength (SINR) to determine if decoding is successful. A subset X of links is *feasible* if there exists a power assignment to the senders such that each link i satisfies the SINR formula $\frac{\sum_{j \in X} I_{ji} + N}{S_i} \leq \beta^{-1}$ within its subset, where S_i is the received signal strength on link i , β and N are fixed constants (dependent on technology and environment), and I_{ji} is the interference strength of link j on link i [15] (see Sec. 5 for full definitions). We can avoid dealing directly with power assignments using a condition for feasibility due to Kesselheim [28] (and shown here to be necessary). The point remains, however, that the feasibility predicate is an asymmetric, cumulative, many-to-one relationship.

Capturing interference with graphs. The aim of this work is to capture the complex feasibility relationship of the physical model with *graphs*, a much more amenable and better studied model. Preferably, a feasible set of links should correspond to an *independent set* in the graph on the links, and vice versa, but since exact capture is impossible, we seek instead approximate representations. Specifically, we want a formulation that constructs a *pair* of graphs that bound feasibility both from below and from above. The two graphs should also be “close” in some sense; specifically we want an independent set in the lower bound graph to induce a low chromaticity subgraph in the upper bound graph. That is, given a set L of links, form graphs $G_l(L)$ and $G_u(L)$ on L such that:

- If S is feasible subset of links, then it is an independent set in $G_l(L)$,
- If S is an independent set in $G_u(L)$, then S is feasible set of links, and
- Chromatic numbers of the subgraphs induced by a subset S are close, or $\max_{S \subseteq L} \frac{\chi(G_u[S])}{\chi(G_l[S])} \leq \rho$.

We may dub the worst ratio ρ over all instances as the *cost* of that graph formulation, the price we pay for using that simpler pairwise and binary graph representation. The least such cost over all graph formulation can then be called the *price of (graph) abstraction*.

Our results. We propose a family of conflict graph representations, parameterized by a sub-linear, non-decreasing function f . It generalizes known families, e.g., disc graphs correspond to linear functions f and pairwise feasibility is captured by constant functions. The graphs in the family have a structural property that allows for effective approximability. We argue that this family captures all meaningful conflict graph representations, modulo constant factors. Our main positive result is that for the right choice of f , our conflict graph representation has a cost of $O(\log^* \Delta)$, where Δ is the ratio between longest and the shortest link length. We also show that all meaningful representations must pay this $\log^* \Delta$ factor. Thus, the price of abstraction, for the SINR model with arbitrary power control, is $\Theta(\log^* \Delta)$. The upper bounds hold for planar instances, and more generally in doubling metrics, while the lower bounds are on the line. We also find that no such results are possible in general metrics nor in terms of the parameter n , the number of links.

We apply our formulations to obtain greatly improved bounds for fundamental wireless scheduling problems. In the link **Scheduling** problem, we want to partition a given set of links into fewest possible feasible sets. In the *weighted capacity* problem, **WCapacity**, the links have associated

positive weight, and we want to find the maximum weight feasible subset. In both cases, our $O(\log^* \Delta)$ -approximations are the first sub-logarithmic approximations known.

Related work. Gupta and Kumar [15] proposed the geometric version of the SINR model, where signal decays as a fixed polynomial of distance; it has since been the default in analytic and simulations studies. They also initiated the average-case analysis of network capacity, giving rise to a large body on “scaling laws”. Moscibroda and Wattenhofer [37] initiated worst-case analysis in the SINR model.

Graph-based models of wireless communication have been very common in the past. Most common are geometric graphs involving circular ranges: *unit-disc graphs* (UDG) have all ranges of the same radius, while in *disc graphs*, the radius can vary with the power assigned, and in the *protocol* model [15] the communication and interference ranges are different. Various attempts have been made to add realism, such as with 2-hop model, or *quasi-unit disc graphs* [2, 33], and the recently studied model of *dual graphs* [32] captures arbitrary unreliability in networks. None of these known models offers though any guarantees on fidelity for representing SINR relationships, see e.g. [38]. Graph formulations for modeling SINR relationships were given previously in [17] followed by [46], but the cost factor was either $O(\log \log \Delta \cdot \log n)$ or $O(\log \Delta)$.

Early work on the **Scheduling** problem includes [5, 7, 4, 13]. NP-completeness results have been given for different variants [13, 27, 34], but as of yet no APX-hardness or stronger lower bounds are known for any related problem in geometric settings, perhaps indicating the difficulty of dealing with the SINR constraints. The related **Capacity** problem, where we seek to find a maximum feasible subset of links, admits constant-factor approximation [28]. This immediately implies a $O(\log n)$ -approximation for **Scheduling**, where n is the number of links. Another approach is to solve links of similar lengths in groups, which results in a $O(\log \Delta)$ -approximation [13, 11, 17]. The question of improved approximation for **Scheduling** has been frequently cited as perhaps the most fundamental open problem in the field [35, 25, 10, 12, 16]. The **WCapacity** problem has applications in several extensions of unit-demand link scheduling problems, such as *stochastic packet scheduling* [44, 41, 30], *general demand vectors*, *multi-path flow* etc. [47]. A more general variant of this problem has been considered in the framework of *combinatorial auctions* [24, 23]. **Scheduling** and **WCapacity** have also been considered with *fixed oblivious power assignments* [21, 8, 17, 19, 9], but the only known sub-logarithmic approximations are known in the case of the *linear power scheme* [19, 45].

Issues of models. A wide range of areas in computer science and mathematics deal with finding simpler abstractions of complex phenomena. Some examples include: a) discrepancy theory, b) dimensionality reductions and embeddings, c) graph augmentations and sandwiching properties, d) graph sparsification, e) curve fitting (including least squares, finite methods, and regression), f) approximation theory (in math), including generalized Fourier series and Chebyshev polynomials, and g) PAC learning.

There are tradeoffs between the accuracy of a model and its complexity of detail. There are legitimate concerns that models and problem formulations are sometimes overly detailed and “brittle”, possibly exceeding reasonable levels of precision (see, e.g., discussion in [39]). The benefits of a coarser model tend to include simpler algorithms, easier analysis, but also less sensitivity to incidental details that may or may not be modelable.

A case in point is the SINR model, which has its issues. Whereas the additivity of interference and the near-threshold nature of signal reception has been borne out in experiments, the geometric decay assumption is far off in essentially all actual environments [43, 36, 42, 14]. One practical alternative is to use facts-on-the-ground in the form of signal strength measurements, instead of the prescriptive distance-based formula [14, 3]. To model that formally, the pessimistic reaction would be to replace the distance assumption with an arbitrary signal-quality matrix, but that runs

into the computational intractability monster, since such a formulation can encode the coloring problem in general graphs [12]. A more moderate approach is to relax the Euclidean assumption to more general metric spaces [9]. The determinacy of the model is another issue. To capture the probabilistic factors observed in the capture of transmissions, one approach is to extend the basic SINR model accordingly, such as with Rayleigh fading; in that case, it has been shown that applying algorithms based on the deterministic formula results in nearly equally good results in that probabilistic setting [6].

Our results suggest that a reassessment of the role of graphs as wireless models might be in order. By paying a small factor (recalling, as well, that $\log^*(x) \leq 5$ in this universe), we can work at higher levels of abstraction, with all the algorithmic and analytical benefits that it accrues. At the same time, hopes for fully constant-factor approximation algorithms for core scheduling problems have receded. It remains to be seen what abstractions are possible for other related settings, especially the case of uniform power.

Roadmap to the rest of the paper. Following the basic definitions in Sec. 2, we derive from first principles what properties link conflict graphs must satisfy (Sec. 3). This can be read independently of the rest of the paper. We next derive (in Sec. 4) two key properties of the family of graphs: how their chromatic numbers relate and how their colorings can be approximated. We then introduce the definitions of the SINR model before starting the technical core of the paper. In Sec. 6, we show that feasibility is captured by two members of our conflict graph family. The implications are discussed in Sec. 7: our main upper bound result, i.e. $O(\log^* \Delta)$ -approximation for **Scheduling** and **WCapacity**; a necessary and sufficient condition for feasibility; and an explicit polynomial-time computable measure of interference. Limitation results are given in Sec. 8, in particular that no better bounds are possible via conflict graphs. For space reasons, only sketches of the proofs are given in this extended abstract.

2 Definitions: Metrics, Functions, Graphs

Communication Links. Consider a set L of n links, numbered from 1 to n . Each link i represents a unit-demand communication request from a sender s_i to a receiver r_i — point-size wireless transmitter/receivers (nodes) located in a metric space with distance function d . We denote $d_{ij} = d(s_i, r_j)$ and refer to $l_i = d(s_i, r_i)$ as the *length* of link i . We let $\Delta(L)$ denote the ratio between the longest and the shortest link lengths in L , and drop L when clear from context. We shall assume in the rest of the paper that all link lengths are distinct, which can be achieved by arbitrarily (but consistently) breaking ties as needed. For sets S_1, S_2 of links, we let $d(S_1, S_2)$ denote the minimum distance between a node in S_1 and a node in S_2 . In particular, we will extensively use $d(i, j) = \min(d_{ij}, d_{ji}, d(s_i, s_j), d(r_i, r_j))$, the minimum distance between nodes on links i, j .

Let $S_i^+ = \{j \in S : l_j > l_i\}$ denote the subset of links in a set S that are longer than link i , and similarly $S_i^- = \{j \in S : l_j < l_i\}$ the subset of links shorter than i .

Doubling Metrics. The *doubling dimension* of a metric space is the infimum of all numbers $\delta > 0$ such that for every ϵ , $0 < \epsilon \leq 1$, every ball of radius $r > 0$ has at most $C\epsilon^{-\delta}$ points of mutual distance at least ϵr where $C \geq 1$ is an absolute constant, and $0 < \epsilon \leq 1$. Metrics with finite doubling dimensions are said to be *doubling*. For example, the m -dimensional Euclidean space has doubling dimension m [22]. We let m denote the doubling dimension of the space containing the links.

Functions. A function f is *sub-linear* if $f(x) = O(x)$. A function f is *strongly sub-linear* if for each constant $c \geq 1$, there is a constant c' such that $cf(x)/x \leq f(y)/y$ for all $x, y \geq 1$ with $x \geq c'y$. Note that if f is strongly sub-linear then $f(x) = o(x)$.

Examples. The functions $f(x) = x^{1-\epsilon}$ for any constant $\epsilon > 0$ and $f(x) = \log x$ are strongly sub-linear¹, while $f(x) = x/\log x$ is not strongly sub-linear even though $f(x) = o(x)$.

Let f be a strongly sub-linear function. For each integer $c \geq 1$, the function $f^{(c)}(x)$ is defined inductively by: $f^{(1)}(x) = f(x)$ and $f^{(c)}(x) = f(f^{(c-1)}(x))$. Let $x_0 = \inf\{x \geq 1, f(x) < x\} + 1$; such a point exists for any $f(x) = o(x)$. The function *iterated* f , denoted $f^*(x)$, is defined by:

$$f^*(x) = \begin{cases} \min_c \{f^{(c)}(x) \leq x_0\}, & \text{if } x > x_0, \\ 1, & \text{otherwise.} \end{cases}$$

Examples. For $f(x) = \log x$, $f^*(x) = \log^* x$ is the well known *iterated logarithm*. It is also easy to check that for $f(x) = \log^{(c)} x$, $f^*(x) = \lceil \frac{\log^* x}{c} \rceil$ and for $f(x) = \sqrt{x}$, $f^*(x) = \lceil \log \log x \rceil$. Note also that $g^*(x) = \Theta(f^*(x))$ if $g(x) = cf(x)$ for a constant $c > 0$.

Graphs. For a graph G and a vertex v , $N_G(v)$ denotes the neighborhood of v in G , i.e., the set of vertices adjacent with v . $\chi(G)$ denotes the chromatic number of graph G , the minimum number of colors needed for a proper (vertex) coloring of G .

A *d-inductive order* of a graph is an arrangement of the vertices from left to right such that each vertex has at most d *post-neighbors*, or neighbors appearing to its right. A *k-simplicial elimination order* is one where the post-neighbors of each vertex can be covered with k cliques. A graph is *d-inductive* (*k-simplicial*) if it has a *d-inductive* (*k-simplicial elimination*) order. It is well known that a *d-inductive* graphs are $d + 1$ -colorable, while the coloring and weighted maximum independent set problems are *k*-approximable in *k-simplicial* graphs [1, 26, 49]. The only inductive or simplicial order we consider for conflict graphs is the *increasing order* of links by length.

3 Formulations of Conflict Graphs

What kind of graphs are conflict graphs? By a “conflict graph formulation” we mean a deterministic rule for forming graphs on top of a set of links. For it to be meaningful as a general purpose mechanism, such a formulation cannot be too context sensitive. We shall postulate some axioms (that by nature should be self-evident) that lead to a compact description of the space of possible conflict graph formulations.

Axiom 1. A *conflict graph formulation* is defined in terms of the pairwise relationship of links.

By nature, graphs represent pairwise relationships; conflict graphs formulations are boolean predicates of pairs of links. More specifically, though, we expect the conflict graph to be defined in terms of the relative standings of the link pairs. That is, the existence of an edge between link i and link j should depend only on the properties of the two links, not on other links in the instance. The only properties of note are the $\binom{4}{2} = 6$ distances between the nodes in the links.

We refer to a *conflict* between two links if the formulation specifies them to be adjacent in the conflict graph; otherwise, they are *conflict-free*.

Axiom 2. A *conflict graph formulation* is independent of positions and scale. *Translating distances or scaling them by a fixed factor does not change the conflict relationship.*

It is an essential feature of the SINR formula (that distinguishes it from other formulations, like unit-disc graphs) is that only relative distances matter. Thus, the positions of the nodes should not matter, only the pairwise distances, and only the relative factors among the distances. There

¹When not otherwise identified, logarithms are base 2.

is a practical limit to which links can truly grow, due to the ambient noise term. However, that only matters when lengths are very close to that limit; we will treat that case separately.

As a result of this axiom, we can factor out the length of the shorter of the two links considered.

Axiom 3. *A conflict formulation is monotonic with increasing distances.*

The reasoning is that a conflict formulation should represent the degree of conflict between pairs of links, or their relative “nearness”. Specifically, if two links conflict and their separation (i.e., one of the distances between endpoints on distinct links) decreases while the links stay of the same length, then the links still conflict. Similarly, if two links are conflict-free and the length of one of them decreases (while their separation stays unchanged), the links stay conflict-free.

Axiom 4. *A conflict formulation should respect pairwise incompatibility. That is, if two (links) cannot coexist in a feasible solution, they should be adjacent in the conflict graph.*

In the case of conflict graphs for links in the SINR model with arbitrary power control, we propose an additional axiom.

Axiom 5. *A conflict formulation for links under arbitrary power control is symmetric with respect to senders and receivers.*

Namely, it should not matter which endpoint of a link is the sender and which is the receiver when determining conflicts. The key rationale for this comes from Kesselheim’s sufficient condition for feasibility, given here as Thm. 4. As we show in Sec. 7.3, this formula is also a necessary condition in doubling metrics, up to constant factors. Thus, feasibility is fully characterized by a symmetric rule (modulo constant factors).

As we shall see, the axioms and the properties of doubling metrics imply that only two distances really matter in the formulation of conflict graphs: the length of the longer link, and the distance between the nearest nodes on the two links (both scaled by the length of the shorter link). This motivates the following definition.

Definition 1. *Let $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a positive non-decreasing sub-linear function. Two links i, j are said to be f -independent if*

$$\frac{d(i, j)}{l_{\min}} > f\left(\frac{l_{\max}}{l_{\min}}\right),$$

where $l_{\min} = \min\{l_i, l_j\}$, $l_{\max} = \max\{l_i, l_j\}$, and otherwise they are f -adjacent. A set of links is f -independent (f -adjacent) if they are pairwise f -independent (f -adjacent), respectively.

Let L be a set of links. The conflict graph $\mathcal{G}_f(L)$ is the graph with vertex set L , where two vertices are adjacent if and only if they are f -adjacent.

Remark. For the constant function $f(x) \equiv \gamma$ for a number γ , we use the notation $\mathcal{G}_\gamma(L)$ for the corresponding conflict graph.

We now argue that all conflict formulations satisfying the above axioms are essentially of the form \mathcal{G}_f , for a function f . They can differ from \mathcal{G}_f but not by more than what can be accounted for by an appropriate constant factor.

Proposition 1. *Every conflict graph formulation \mathcal{K} is captured by \mathcal{G}_f , for some non-decreasing and sub-linear function f . Namely, there is a constant γ such that \mathcal{K} is sandwiched by \mathcal{G}_f and $\mathcal{G}_{\gamma f}$, i.e., $\mathcal{G}_f(L) \subseteq \mathcal{K}(L) \subseteq \mathcal{G}_{\gamma f}(L)$, for every link set L .*

Proof. By Axiom 1, a conflict formulation \mathcal{K} is a function of link pairs, more specifically, the distances among the four points. By Axiom 2, we can use normalized distances, and will choose to factor out the length of the shorter link. By Axiom 5, it does not matter which of them involve senders and which involve receivers.

Now, consider two links $i = (s_i, r_i)$ and $j = (s_j, r_j)$, where $l_i \leq l_j$, and assume without loss of generality that s_i and s_j are the nearest points on the two links. Let us denote for short $d = d(i, j) = d(s_i, s_j)$. We aim to show that decisions regarding adjacency in \mathcal{K} can be determined in terms of constant multiples of d and l_j .

First, recall that by Axiom 4, pairwise incompatible links must be adjacent in any conflict graph. Thus, as derived in Thm. 6, we may restrict attention to the case that $d(i, j) \geq cl_i$, for an absolute constant c . In that case, it follows that distance $d(r_i, s_j)$ is at most constant times the distance from i to j , i.e., $d(s_i, s_j) \leq d(r_i, s_j) \leq (1 + 1/c)d(s_i, s_j)$.

Next, we claim that $d(s_i, r_j)$ is within a constant multiple of $q = \max(d, l_j/2)$. By definition of d , it holds that $d(s_i, r_j) \geq d$, while by the triangular inequality, $d(s_i, r_j) \geq l_j - d(s_i, s_j) = l_j - d$. Thus, $d(s_i, r_j) \geq \max(d, l_j - d) \geq q$. Also, by triangular inequality, $d(s_i, r_j) \leq d + l_j \leq 3q$.

Finally, for by triangular inequality, $d(r_i, r_j) \leq d + l_i + l_j \leq d + 2l_j$, and $d(r_i, r_j) \geq d - l_i - l_j \geq d - 2l_j$. So, defining $q' = \max(d, l_j/4)$, we similarly obtain that $q' \leq d(r_i, r_j) \leq 9q'$. It follows that all the five distances between endpoints are within constant multiples of $d(i, j)$ and l_j , relative to the shorter link length l_i .

Hence, by monotonicity (Axiom 3), \mathcal{K} is dominated by a conflict graph formulation \mathcal{H} defined by a monotone boolean predicate of two variables: length of the longer link l_j , and the distance $d(i, j)$ between the links (scaled by the shorter link). But, an arbitrary monotone boolean predicate of two variables x, y can be represented by a relationship of the form $y > f(x)$, for some function f . Thus, \mathcal{K} is dominated by \mathcal{G}_f , for some non-decreasing function f . Also, by the same arguments, \mathcal{K} dominates \mathcal{G}_{cf} for a constant c .

Finally, sub-linearity is a necessary condition, since super-linear growth would break Axiom 3. \square

4 Properties of Conflict Graphs

We explore two types of properties of conflict graphs. The first type is concerned with gaps between the chromatic numbers of conflict graphs, or the relative difference of the chromatic numbers of graphs \mathcal{G}_f and $\mathcal{G}_{f'}$. We show that the introduction of f increases the chromatic number of \mathcal{G}_γ by a rather small factor depending on f . This is a key result that will be used to derive the approximation factor in the main result of this paper. We also show that the introduction of constant factor γ changes the chromatic number of \mathcal{G}_f only by a constant factor.

In the second part we consider algorithmic properties of graphs \mathcal{G}_f . In particular, we prove that graphs \mathcal{G}_f are constant-simplicial. Thus, constant factor approximation algorithms for vertex coloring, weighted maximum independent set and several other \mathcal{NP} -hard problems follow. This allows us to algorithmically approximate feasibility with graphs.

4.1 Gaps Between Chromatic Numbers of Conflict Graphs

We start by showing that the difference between the chromatic numbers of \mathcal{G}_γ and $\mathcal{G}_{\gamma f}$ is a factor of at most $O(f^*(\Delta))$, where f^* is the iterated f function. This result is obtained by proving that for any independent set S in $\mathcal{G}_\gamma(L)$, the graph $\mathcal{G}_{\gamma f}(S)$ is $O(f^*(\Delta))$ -inductive. To this end, we want to show that, for any given link i in S , any set T of mutually γ -independent links in S_i^+ that are γf -neighbors of i is small, or $O(f^*(\Delta))$. We do so by showing that the progression of lengths of

the links in T must be fast growing, or inversely with f ; the number of links must therefore be bounded by the iterated f function.

Theorem 1. *For any set of links L , a constant $\gamma \geq 1$ and a non-decreasing strongly sub-linear function f , $\chi(\mathcal{G}_{\gamma f}(L)) = O(f^*(\Delta)) \cdot \chi(\mathcal{G}_{\gamma}(L))$.*

Proof. Let S be a γ -independent set in L . Consider any link $i \in S$ and let T denote the set of links in S_i^+ that are f -adjacent with i . We will show that $|T| = O(f^*(\Delta))$. Note that for each $j \in T$, $d(i, j) \leq \gamma l_i f(l_j/l_i)$. Let p_j denote the endpoint of link $j \in T$ closest to link i . We split T into two subsets T_1 and T_2 , where

$$T_1 = \{j \in T : d(p_j, r_i) \leq \gamma l_i f(l_j/l_i)\} \text{ and } T_2 = T \setminus T_1 \subseteq \{j \in T : d(p_j, s_i) \leq \gamma l_i f(l_j/l_i)\}.$$

Let us first consider T_1 . Let $j, k \in T_1$ be two links with $l_k < l_j$. Then,

$$d(p_j, r_i) \leq \gamma l_i f(l_j/l_i), \quad (\text{because } j, k \in T_1) \quad (1)$$

$$d(p_k, r_i) \leq \gamma l_i f(l_k/l_i), \quad (2)$$

$$d(p_j, p_k) \geq d(j, k) > \gamma l_k, \quad (j, k \text{ are } \gamma\text{-independent}) \quad (3)$$

By plugging the inequalities above into the triangle inequality $d(p_j, p_k) \leq d(p_j, r_i) + d(r_i, p_k)$, we obtain $\gamma l_k < \gamma l_i f(l_j/l_i) + \gamma l_i f(l_k/l_i) \leq 2\gamma l_i f(l_j/l_i)$, where the last inequality follows from the assumption that $l_k < l_j$ and that f is a non-decreasing function. Thus,

$$l_k/l_i < 2f(l_j/l_i). \quad (4)$$

Denote $g(x) \equiv 2f(x)$. Note that $g(x)$ is strongly sub-linear; hence, there exists $x_0 = \inf\{x \geq 1, g(x) < x\} + 1$. Let $1, 2, \dots, t = |T_1|$ be the arrangement of the links in T_1 in increasing order by length and let $\lambda_j = \frac{l_j}{l_i}$ for $j = 1, 2, \dots, t$. Let h be the link with the smallest index such that $\lambda_h \geq x_0$. We will bound the number of links in $A = \{1, 2, \dots, h-1\}$ and $B = \{h, h+1, \dots, t\}$ separately. $|A|$ can be bounded by a simple application of the doubling property of the space. Note that for all $j \in A$, $f(l_j/l_i) \leq f(x_0) = O(1)$ because $l_j/l_i < x_0$. Thus, the system of inequalities (1-3) implies that the mutual distance between different points p_j with $j \in A$ is at least γl_i , while their distance from r_i is at most $\gamma f(x_0) l_i$; hence, we have that $|A| = O(f(x_0)^m) = O(1)$.

Now let us bound $|B|$, using (4). We have that

$$x_0 \leq \lambda_h < g(\lambda_{h+1}) \leq g(g(\lambda_{h+2})) \leq \dots \leq g^{(t-h)}(\lambda_t),$$

which implies that $t - h \leq g^*(\lambda_t) = O(f^*(\Delta))$. Recall that $h = |A| = O(1)$; hence, $t = O(f^*(\Delta))$.

This completes the proof that $|T_1| = O(f^*(\Delta))$. The set T_2 is handled similarly. In this case, for any pair of links $j, k \in T_2$ with $l_j > l_k$, the system of inequalities (1-3) holds with r_i replaced with s_i ; the rest of the argument is identical. These results imply the theorem. \square

Next we show that the chromatic number of \mathcal{G}_f is of the same order as the chromatic number of $\mathcal{G}_{\gamma f}$. We prove that any independent set S in \mathcal{G}_f is constant-inductive as an induced subgraph of $\mathcal{G}_{\gamma f}$. To this end, we prove, using the strong sub-linearity of f (Lemma 1), that for any link $i \in S$, the set of neighbors of i in S_i^+ mainly consists of links of length $\Theta(l_i)$. The number of such links is bounded by a rather straightforward application of the doubling property of the space, using the fact that those links form an independent set, while at the same time are adjacent with link i .

Lemma 1. *Let f be a non-decreasing strongly sub-linear function and i, j, k be links. If links j, k are longer than i , are f -independent and are γf -adjacent with i , then $\min\{l_j, l_k\} \leq c l_i$, where the constant c depends only on function f and constant γ .*

Proof. Assume without loss of generality that $l_k < l_j$. Since f is strongly sub-linear, there is a constant $c > 0$ such that $2\gamma f(x)/x \leq f(y)/y$ whenever $x \geq cy$. We will show that $l_k \leq cl_i$. Let us assume, for contradiction, $l_k > cl_i$. Let p_j (p_k) denote the endpoint of link j (link k) closest to link i . Then,

$$d(p_j, r_i) \leq \gamma l_i f(l_j/l_i), \quad (j, k \text{ are } \gamma f\text{-adjacent with } i) \quad (5)$$

$$d(p_k, r_i) \leq \gamma l_i f(l_k/l_i), \quad (6)$$

$$d(p_j, p_k) \geq d(j, k) > l_k f(l_j/l_k), \quad (j, k \text{ are } f\text{-independent}). \quad (7)$$

By plugging these inequalities into the triangle inequality we get:

$$l_k f(l_j/l_k) < d(p_j, p_k) \leq d(p_j, r_i) + d(r_i, p_k) \leq \gamma l_i f(l_j/l_i) + \gamma l_i f(l_k/l_i) \leq 2\gamma l_i f(l_j/l_i).$$

Let us denote $x = l_j/l_i$ and $y = l_k/l_i$. Note that $x > cy$. The inequality above asserts that $2\gamma f(x)/x > f(y)/y$, which contradicts the definition of c . This completes the proof. \square

Theorem 2. *Let L be a set of links and f be a non-decreasing strongly sub-linear function. Then $\chi(\mathcal{G}_f(L)) = \Theta(\chi(\mathcal{G}_{\gamma f}(L)))$ for any constant $\gamma > 0$.*

Proof. We assume w.l.o.g. that $\gamma \geq 1$. Thus, we only need to show that $\chi(\mathcal{G}_{\gamma f}(L)) = O(\chi(\mathcal{G}_f(L)))$. Consider any independent set S in $\mathcal{G}_f(L)$. It suffices to show that $\mathcal{G}_{\gamma f}(S)$ is constant-inductive. Let us fix a link i and let $T = S_i^+ \cap N_{\mathcal{G}_{\gamma f}(S)}(i)$ denote the set of neighbors of i in $\mathcal{G}_{\gamma f}(S)$ that are longer than i . It is enough to show that $T = O(1)$. Recall that all the links in T are γf -adjacent with i and are f -independent among each other. By applying Lemma 1 for all pairs $j, k \in T$, we conclude that there is a constant c such that all the links in T , except perhaps one, have length at most cl_i . Let T' be the subset of T containing those links. It is enough to show that $|T'| = O(1)$. For each link $j \in T'$, let p_j denote the endpoint of j closest to link i . We split T' into two subsets T'_1 and T'_2 , where

$$T'_1 = \{j \in T' : d(p_j, r_i) \leq \gamma l_i f(l_j/l_i)\} \text{ and } T'_2 = T' \setminus T'_1 \subseteq \{j \in T' : d(p_j, s_i) \leq \gamma l_i f(l_j/l_i)\}.$$

We first bound $|T'_1|$. Note that for each pair of links $j, k \in T'_1$ with $l_j > l_k$, the distance $d(p_j, p_k)$ is at least $d(p_j, p_k) > l_k f(l_j/l_k) \geq f(1)l_i$ because j, k are f -independent. On the other hand, for each $j \in T'_1$, the distance $d(p_j, r_i)$ is at most $d(p_j, r_i) \leq \gamma l_i f(l_j/l_i) \leq \gamma f(c)l_i$ because i, j are γf -adjacent and $l_j \leq cl_i$. We conclude that $|T'_1| = O((\gamma f(c)/f(1))^m) = O(1)$, using the doubling property of the metric space.

We can prove that $|T'_2| = O(1)$ in a similar manner, by replacing r_i with s_i in the formulas. These results imply the theorem. \square

4.2 Algorithmic Properties of Conflict Graphs

We prove that every conflict graph \mathcal{G}_f with strongly sub-linear function f is constant-simplicial. This guarantees, among other properties, that the vertex coloring and maximum weighted independent set problems in these graphs can be efficiently approximated within constant factors [1, 26, 49].

We give a simple argument that holds in the plane, where it holds for essentially all sub-linear functions. It is based on splitting the plane into 60° sectors emanating from a given node of a link, and arguing that all adjacent longer links within a sector must form a clique. With a more detailed argument, the result can be extended to general doubling metrics, but requires then strong sub-linearity.

Proposition 2. *Let f be a non-decreasing function such that $f(x)/x$ is non-increasing and let L be a set of links in the plane. Then $\mathcal{G}_f(L)$ is 12-simplicial.*

Proof. Consider a link i and let $S \subseteq L_i^+$ be the longer neighbors of i in $\mathcal{G}_f(L)$. We partition S into two sets: S_1 , with links that are closer to the sender node s_i , and S_2 , the links that are closer to r_i . Consider S_1 first. For a link $j \in S_1$, let p_j denote the endpoint of j that is the closest to s_i , i.e., $d(i, j) = d(p_j, r_i)$. Then we have, from f -dependence, that $d(p_j, s_i) \leq l_i \cdot f(l_j/l_i)$ for all $j \in S_1$. Partition the plane into six 60° sectors emanating from s_i . Let j, k be two links such that $l_k > l_j$ and p_j and p_k fall in the same sector. By considering the triangle (p_j, p_k, s_i) , we can see that the edge $p_j p_k$ must be no longer than $\max\{d(p_j, s_i), d(p_k, s_i)\}$. Thus,

$$d(j, k) \leq d(p_j, p_k) \leq l_i \cdot f(l_k/l_i) \leq l_j \cdot f(l_k/l_j),$$

where the second inequality follows because $f(x)$ is non-decreasing and the third one follows because $f(x)/x$ is non-increasing. This shows that the links j, k must be adjacent. Thus, S_1 can be covered with at most 6 cliques. An identical argument holds for S_2 , by splitting the plane around r_i . This completes the proof. \square

Remark. The proof above cannot be replicated in general doubling metrics. Moreover, it can be shown that there are functions f such that \mathcal{G}_f is constant-simplicial in the plane, but is not so even in a one dimensional doubling metric. The linear function $f(x) = x$ is such an example. This claim can be demonstrated by the following example with $n + 1$ points p_0, p_1, \dots, p_n , where $d(p_i, p_0) = 2^i$ for all $i \geq 1$ and $d(p_i, p_k) = 2^i + 1$ for all $i > k \geq 1$. It is straightforwardly checked that the points in this example form a 1-dimensional doubling space. On the other hand, the f -neighborhood of point p_0 contains a set of $\log n$ f -independent points. A similar example was also observed in [48] in a different context. However, we prove that \mathcal{G}_f is constant-simplicial in doubling metrics for *strongly sub-linear* functions f .

We show as before, that for a link i , the set of longer links adjacent with i can be covered with a small number of cliques. We start by showing (using Lemma 1) that the neighbors of i of length $\Omega(l_i)$ must form a single clique. The rest of the links (having length $O(l_i)$) can be covered by cliques using a simple clustering procedure. We show that the cluster-heads must be separated by a distance $\Omega(l_i)$, which implies, using the doubling property of the space and the fact that all the links in consideration are neighbors of link i and have length $O(l_i)$, that the number of clusters (cliques) must be bounded by a constant.

Theorem 3. *Let f be a non-decreasing strongly sub-linear function with $f(x) \geq 1$ for all $x \geq 1$. Then for each set L , $\mathcal{G}_f(L)$ is constant-simplicial.*

Proof. Let $i \in L$ be any link and let the subset $S \subseteq L_i^+$ consist of links in L longer than i and f -adjacent with i . We show that S can be covered with a constant number of cliques.

Let $j, k \in S$ be any pair of f -independent links in S . By Lemma 1, there is a constant $c > 0$ depending only on f , such that at least one of the links j, k is not longer than cl_i . Thus, the set of links $j \in S$ with $l_j > cl_i$ forms a single clique.

It remains to show that the subset of S with links of length at most cl_i can be covered with a constant number of cliques. Let T denote this subset.

We split T into a set of cliques by the following procedure. Pick an arbitrary link $j \in T$. Let N_j^1 denote the set of links k (including link j) with $\min\{d(s_j, s_k), d(s_j, r_k)\} \leq l_i/2$ and N_j^2 denote the set of links k with $\min\{d(r_j, s_k), d(r_j, r_k)\} \leq l_i/2$. Set $T = T \setminus (N_j^1 \cup N_j^2)$. Repeat until T is empty. Let R be the links picked by the procedure above.

<i>Notation</i>	<i>Meaning</i>	<i>Topic</i>	<i>Page</i>
$f^{(c)}(x)$	function f applied c times	<i>Functions</i>	4
$f^*(x)$	iterated f		4
$\widehat{\log}(x)$	$\approx \log^{2/(\alpha-m)}(x)$		11
m	the doubling dimension of the metric space	<i>Metric Space</i>	3
d	the distance function of the metric space		3
n	the number of links	<i>Links</i>	3
s_i, r_i	sender and receiver nodes of link i		3
l_i	the length of link i , $l_i = d(s_i, r_i)$		3
d_{ij}	the distance from s_i to r_j , $d_{ij} = d(s_i, r_j)$		3
$d(i, j)$	the minimum distance between links i, j		3
$\Delta(L)$	the maximum ratio between link lengths in L		3
L_i^+	the subset of links of L longer than link i		3
L_i^-	the subset of links of L shorter than link i		3
$\chi(G)$	the chromatic number of graph G	<i>Graphs</i>	4
$N_G(v)$	the neighborhood of vertex v in graph G		4
$\mathcal{G}_f(L)$	the f -adjacency graph over the set L		5
$\mathcal{G}_\gamma(L)$	the f -adjacency graph over L with $f(x) = \gamma$		5
P	power assignment, $P : L \rightarrow \mathbb{R}_+$	<i>SINR</i>	10
α	the path loss exponent		11
β	the SINR threshold value		11
N	the ambient noise term		11
$OPTS(L)$	the optimum schedule length of set L		15
I	the influence operator		11
$I(L)$	same as $\max_{i \in L} I(L_i^-, i)$		11

Table 1: Notations.

Note that for each $j \in R$, N_j^1 and N_j^2 are cliques. Indeed, consider e.g. N_j^1 and let $k_1, k_2 \in N_j^1$ and $l_{k_1} < l_{k_2}$. The triangle inequality and the assumption $f(x) \geq 1$ for $x \geq 1$ imply that $d(k_1, k_2) \leq l_i < l_{k_1} f(l_{k_2}/l_{k_1})$, which means that k_1, k_2 are f -adjacent. Moreover, these cliques cover T . Let us show that $|R| = O(1)$. For each $j \in R$, let p_j denote the endpoint of j that is closest to i . We split R into two subsets, R_1 and R_2 (based on the fact that R consists of links f -adjacent with i), where

$$R_1 = \{j \in R : d(p_j, r_i) \leq l_i f(l_j/l_i)\} \text{ and } R_2 = R \setminus R_1 \subseteq \{j \in R : d(p_j, s_i) \leq l_i f(l_j/l_i)\}.$$

Let us consider R_1 first. Recall that for each $j \in R$, $l_j \leq cl_i$. From the definition of R_1 we have that for each $j \in R_1$, $d(p_j, r_i) \leq l_i f(l_j/l_i) \leq l_i f(c)$, i.e., the points p_j are inside the ball of radius $l_i f(c)$ centered at s_i . On the other hand, we have by the construction of R that for any $j, k \in R$, $d(p_j, p_k) \geq d(j, k) > l_i/2$. Hence, by applying the doubling property of the metric space we get that $|R_1| \leq (l_i f(c)/(l_i/2))^m = (2f(c))^m = O(1)$. A similar argument holds for R_2 , by replacing r_i with s_i . Thus, we get $|R| = O(1)$, which completes the proof. \square

5 Definitions: SINR, Feasibility

SINR Model and Feasibility. A *power assignment* for a set L of links is a function $P : L \rightarrow \mathbb{R}_+$. For each link i , $P(i)$ defines the power level used by the sender node s_i . In the *physical model* (or

SINR model) of communication [40], when using a power assignment P , a transmission of a link i is successful if and only if

$$\frac{P(i)}{l_i^\alpha} \geq \beta \cdot \left(\sum_{j \in S \setminus \{i\}} \frac{P(j)}{d_{ji}^\alpha} + N \right), \quad (8)$$

where N is a constant denoting the ambient noise, β denotes the minimum SINR (Signal to Interference and Noise Ratio) required for a message to be successfully received, $\alpha \in (2, 6)$ is the path loss constant and S is the set of links transmitting concurrently with link i . Here the left side of the inequality is interpreted as the received signal power of link i and the sum on the right side is interpreted as the interference on link i caused by concurrently transmitting links.

A set S of links is called *P-feasible* if the condition (8) holds for each link $i \in S$ when using power P . We say S is *feasible* if there exists a power assignment P for which S is *P-feasible*. Similarly, a collection of sets is *P-feasible/feasible* if each set in the collection is. Note that we do not assume limits on the available power, which means that the noise term can be ignored. The case of a maximum power limit requires primarily that the links that are close to maximum length be handled separately using the maximum power available [29], something that remains to be studied.

The Influence Operator and a Sufficient Condition for Feasibility. The *influence operator* I is defined as follows. For links i, j , let $I(i, j) = \frac{l_i^\alpha}{d(i, j)^\alpha}$ and define $I(i, i) = 0$ for simplicity of notation. The operator I is additively expanded: for a set S of links and a link i , let $I(S, i) = \sum_{j \in S} I(j, i)$ and $I(i, S) = \sum_{j \in S} I(i, j)$. We will use the notation $I(L) = \max_{i \in L} I(L_i^-, i)$.

In order to identify feasible sets, we will use the following sufficient condition for feasibility.

Theorem 4. [28] *For any set of links L in a metric space, if $I(L) < \frac{1}{2 \cdot 3^\alpha (4\beta + 2)}$, then L is feasible.*

Sensitivity of Feasible Sets. A set of links is called *p-P-feasible* if it is *P-feasible* with the parameter β replaced with number p . The following sensitivity argument has proved useful. It shows, in particular, that constant factor changes to the threshold parameter β do not affect asymptotic results by more than a constant factor.

Theorem 5. [18] *Let p, p' be positive values, P be a power assignment, and L be a p - P -feasible set. Then L can be partitioned into $\lceil 2p'/p \rceil$ sets each of which is p' - P -feasible.*

Fading Metrics. *Fading metrics* are doubling metrics with doubling dimension $m < \alpha$. We shall assume, without stating so explicitly, that the links are located in a fading metric.

6 Capturing Feasibility with Conflict Graphs

We show that for appropriate constant $\gamma > 0$ and function f , SINR-feasibility is “trapped” between graph representations \mathcal{G}_γ and \mathcal{G}_f ; namely, each feasible set is an independent set in $\mathcal{G}_\gamma(L)$ and each independent set in $\mathcal{G}_f(L)$ is feasible. In particular, this holds for $f(x) = \gamma' \widehat{\log}(x)$ for an appropriate constant $\gamma' > 0$, where the function $\widehat{\log}(x)$ is defined for $x \geq 1$ by $\widehat{\log}(x) = \max(\log^{2/(\alpha-m)}(x), 1)$. The gap between these approximations is quantified using our results in Sec. 4.1, ultimately leading to $O(\log^* \Delta)$ approximation for scheduling problems.

6.1 Independence of Feasible Sets

The theorem below is based on the simple observation that two links in the same “highly feasible” set must be spatially separated by at least a multiple of the length of the shorter link, implying γ -independence for some $\gamma > 0$. The constant γ may then be adapted using Thm. 5, i.e. a feasible set can be split into a constant number of γ' -independent sets for any constant $\gamma' > 0$.

Theorem 6. *For any constant $\gamma > 0$, a $(\gamma + 1)^\alpha$ -feasible set is γ -independent. In particular, if $\beta > 1$ then each feasible set is $(\beta^{1/\alpha} - 1)$ -independent.*

Proof. It suffices to show that two links in the same $(\gamma + 1)^\alpha$ -feasible set must be γ -independent. Let i, j be such links. Since i, j are in the same $(\gamma + 1)^\alpha$ -feasible set, the SINR condition implies that there is a power assignment P such that:

$$P(i)/l_i^\alpha > (\gamma + 1)^\alpha P(j)/d_{ji}^\alpha \text{ and } P(j)/l_j^\alpha > (\gamma + 1)^\alpha P(i)/d_{ij}^\alpha.$$

By multiplying together the inequalities above, canceling $P(i)$ and $P(j)$ and raising to the power of $1/\alpha$, we obtain:

$$d_{ij}d_{ji} > (\gamma + 1)^2 l_i l_j. \quad (9)$$

Let us show first that $\min\{d_{ij}, d_{ji}\} > \gamma \min\{l_i, l_j\}$. Indeed, if the opposite was true, e.g. if $d_{ij} \leq \gamma \min\{l_i, l_j\}$, the triangle inequality would imply that $d_{ji} \leq d_{ij} + l_i + l_j \leq (\gamma + 2) \max\{l_i, l_j\}$, which would contradict to (9): $d_{ij}d_{ji} \leq \gamma(\gamma + 2)l_i l_j \leq (\gamma + 1)^2 l_i l_j$.

Now consider $d(s_i, s_j)$. Let us assume, for contradiction, that e.g. $d(s_i, s_j) \leq \gamma l_i \leq \gamma l_j$. Then the triangle inequality would imply $d_{ji} \leq d(s_i, s_j) + l_i \leq (\gamma + 1)l_i$ and $d_{ij} \leq d(s_i, s_j) + l_j \leq (\gamma + 1)l_j$, which would again yield a contradiction to (9). We prove in the same manner that $d(r_i, r_j) > \gamma \min\{l_i, l_j\}$ and conclude that $d(i, j) > \gamma \min\{l_i, l_j\}$, i.e., i and j are γ -independent. \square

6.2 Feasibility of Independent Sets

Here we show that for a large enough constant $\gamma > 0$, $\widehat{\gamma \log}$ -independence implies feasibility. In particular, we show that if a set S is $\widehat{\gamma \log}$ -independent then $I(S) = O(\gamma^{m-\alpha})$. Since we assumed that $m < \alpha$, an appropriate choice of γ yields feasibility via Thm. 4.

The argument consists of the following stages. For any given link $i \in S$, we first split S_i^- into length classes, or *equilength subsets*, where each equilength subset contains links differing by at most a factor of 2 in length. We bound the influence on link i for each of those subsets separately, and then combine those bounds using the additivity of the influence operator I .

For each equilength subset S the following common technique is applied: partition the plane into concentric annuli around the link i , count the number of links in each annulus and bound $I(S_i^-, i)$ based on these numbers and the fact that the links within the same annulus have almost the same influence on link i (because they are at roughly the same distance from i and have roughly similar lengths). The number of links in each annulus can be bounded using the doubling property of the space and independence of the links. The influence bound obtained for each subset S is $O((\gamma \widehat{\log}(l_i/\ell))^{m-\alpha})$, where ℓ is the longest link length in S . The function $\widehat{\log}$ is chosen so that combining those bounds in a sum results in an upper bound of $I(S_i^-, i) = O(\gamma^{m-\alpha})$.

We will use the following two technical observations.

Fact 1. *Let $\alpha \geq 1$ and $r \geq 0$ be real numbers. Then $\frac{1}{r^\alpha} - \frac{1}{(r+1)^\alpha} \leq \frac{\alpha}{(r+1)^{\alpha+1}}$.*

Fact 2. *Let $g(x) = \frac{1}{(q+x)^\delta}$, where $\delta > 1$ and $q \geq 1$. Then $\sum_{r=0}^{\infty} g(r) = O(q^{1-\delta})$.*

The following lemma bounds the influence of an equilength 1-independent set S on a long link i that is f -independent from the set S . This will be the main building block to be used for showing that $\gamma \log$ -independent sets are feasible. The proof uses the annuli argument mentioned above.

Lemma 2. *Let f be a non-decreasing function, such that $f(x) \geq 1$ whenever $x \geq 1$. Let S be an equilength 1-independent set of links, and let i be a link s.t. for each $j \in S$, $l_i \geq l_j$ and i and j are f -independent. Then $I(S, i) = O((f(l_i/\ell))^{m-\alpha})$, where ℓ denotes the longest link length in S .*

Proof. Let us denote $q = f(l_i/\ell)$. Note that $q \geq 1$ because $l_i/\ell \geq 1$.

Let us split S into two subsets S' and S'' , where S' contains the links of S that are closer to r_i than to s_i , i.e., $S' = \{j \in S : \min\{d(s_j, r_i), d(r_j, r_i)\} \leq \min\{d(s_j, s_i), d(r_j, s_i)\}\}$ and $S'' = S \setminus S'$. Let us bound $I(S', i)$ first.

For a link $j \in S'$, let p_j denote the endpoint of link j that is closest to r_i , i.e., $d(i, j) = d(p_j, r_i)$. Consider the “chain” of subsets $S_1 \subseteq S_2 \subseteq \dots \subseteq S'$, where

$$S_r = \{j \in S' : d(j, i) = d(p_j, r_i) \leq q\ell/2 + (r-1)\ell/2\}.$$

Let $M_r \geq \max_{j \in S_r \setminus S_{r-1}} I(j, i)$ be some upper bound on the maximum of $I(j, i)$ in the annulus $S_r \setminus S_{r-1}$ for $r = 2, 3, \dots$. The value $I(S', i)$ can be bounded as follows:

$$\begin{aligned} I(S', i) &= I(S_1, i) + \sum_{r \geq 2} \sum_{j \in S_r \setminus S_{r-1}} I(j, i) \\ &\leq I(S_1, i) + \sum_{r \geq 2} M_r \cdot |S_r \setminus S_{r-1}| \\ &= I(S_1, i) + \sum_{r \geq 2} M_r (|S_r| - |S_{r-1}|) \\ &= I(S_1, i) - |S_1| M_2 + \sum_{r \geq 2} |S_r| (M_r - M_{r+1}), \end{aligned} \tag{10}$$

where the last line follows by a simple rearrangement of the sum. We will next bound the sizes of subsets S_r and find bounds M_r .

Claim 1. $S_1 = \emptyset$.

Proof. For each link $j \in S'$, $d(p_j, r_i) = d(i, j) > l_j q \geq \ell q/2$ because i and j are f -independent and S is an equilength set with maximum link length ℓ and minimum link length at least $\ell/2$. \square

Claim 2. For each $r \geq 2$, $|S_r| \leq C(q + r - 1)^m$, where C is an absolute constant.

Proof. We bound $|S_r|$ using the doubling property of the metric space. Consider any $j, k \in S_r$ such that $l_j \geq l_k$. By the assumption, j, k are 1-independent; hence, $d(p_j, p_k) \geq d(j, k) > \min\{l_j, l_k\} \geq \ell/2$. By the definition of S_r , $d(p_j, r_i) \leq q\ell/2 + (r-1)\ell/2$ for each $j \in S_r$. Because the metric space has doubling dimension m , the number of points p_j with $j \in S_r$ (hence, also the size $|S_r|$) can be bounded as follows:

$$|S_r| = |\{p_j\}_{j \in S_r}| < C \cdot \left(\frac{q\ell/2 + (r-1)\ell/2}{\ell/2} \right)^m = C(q + r - 1)^m.$$

\square

Claim 3. Let $M_r = \frac{2^\alpha}{(q+r-2)^\alpha}$. For each $r \geq 2$, $\max_{j \in S_r \setminus S_{r-1}} \{I(j, i)\} < M_r$.

Proof. For each $r > 1$ and for any link $j \in S_r \setminus S_{r-1}$, we have that $l_j \leq \ell$ and $d(i, j) > q\ell/2 + (r - 2)\ell/2$; hence, $I(j, i) = \frac{l_j^\alpha}{d(i, j)^\alpha} < \left(\frac{\ell}{q\ell/2 + (r-2)\ell/2} \right)^\alpha = \frac{2^\alpha}{(q+r-2)^\alpha}$. \square

By Claim 1, the first two terms of (10) are zero. Let us fix any $r \geq 2$. Let M_r be as in Claim 3. By Fact 1, $M_r - M_{r+1} \leq \alpha 2^\alpha / (q + r - 1)^{\alpha+1}$, and by Claim 2,

$$|S_r|(M_r - M_{r+1}) < \frac{C\alpha 2^\alpha (q + r - 1)^m}{(q + r - 1)^{\alpha+1}} = \frac{C\alpha 2^\alpha}{(q + r - 1)^{\alpha-m+1}}.$$

By plugging these inequalities into (10) and using Fact 2, we get the desired bound for $I(S', i)$:

$$I(S', i) < \sum_{r \geq 2} |S_r|(M_r - M_{r+1}) < C\alpha 2^\alpha \sum_{r \geq 2} \frac{1}{(q + r - 1)^{\alpha-m+1}} \in O(q^{m-\alpha}).$$

The proof holds symmetrically for the set S'' . Recall that S'' consists of the links of S that are closer to the sender s_i than to the receiver r_i . Now, we can define the set $\{p_j\}_{j \in S''}$ where p_j is the endpoint of link j that is closest to r_i , for each $j \in S''$. The rest of the proof will be identical, by replacing r_i with s_i in the formulas. \square

Having a bound for the influence of each equilength set, we can now split the whole set into equilength subsets (length classes), bound the influence of each equilength subset using Lemma 2 and combine them into a series that converges when we choose $f(x) = \gamma \widehat{\log}(x)$.

Theorem 7. *Let L be a $\gamma \widehat{\log}$ -independent set with $\gamma \geq 1$. Then $I(L) = O(\gamma^{m-\alpha})$.*

Proof. Let us fix an arbitrary link $i \in L$. We have for each $j \in L_i^-$, $d(i, j) > \gamma l_j \widehat{\log}(l_i/l_j)$ because of $\gamma \widehat{\log}$ -independence and that $l_i \geq l_j$. Let ℓ_0 denote the minimum link length in L_i^- . We partition L_i^- into at most $\lceil \log l_i/\ell_0 \rceil$ equilength subsets L_1, L_2, \dots as follows:

$$L_t = \{j \in L_i^- : 2^{t-1}\ell_0 \leq l_j < 2^t\ell_0\},$$

for $t = 1, 2, \dots$. Let ℓ_t be the longest link length in L_t . The conditions of Lemma 2 hold for each L_t : it is an equilength 1-independent set ($\gamma \widehat{\log}$ -independence implies 1-independence for $\gamma \geq 1$) and is f -independent from link i , with $f = \gamma \widehat{\log}$. Note also that $f(x) \geq 1$ when $x \geq 1$. Applying the lemma, we obtain

$$I(L_t, i) = O\left((\gamma \widehat{\log}(l_i/\ell_t))^{m-\alpha}\right).$$

Let d denote the largest index t for which L_t is not empty. By the definition of function $\widehat{\log}$ we have that $\widehat{\log}(l_i/\ell_d) \geq 1$ and for each $t < d$, $\widehat{\log}(l_i/\ell_t) = \log^{2/(\alpha-m)}(l_i/\ell_t) \geq (d-t)^{2/(\alpha-m)}$. Thus,

$$I(L_i^-, i) = \sum_{t=1}^d I(L_t, i) \leq c\gamma^{m-\alpha} \left(1 + \sum_{t=1}^d \left((d-t)^{2/(\alpha-m)}\right)^{m-\alpha}\right) = O(\gamma^{m-\alpha}),$$

where c is a constant. Since this holds for arbitrary $i \in L$, we have that $I(L) = O(\gamma^{m-\alpha})$. \square

Since the theorem above holds for any $\gamma \geq 1$, we obtain the desired result.

Corollary 1. *There is a constant $\gamma \geq 1$ such that each $\gamma \widehat{\log}$ -independent set is feasible.*

7 Implications

7.1 Scheduling and WCapacity Approximation

Using our method of capturing feasibility with graphs, we approximate **Scheduling** and **WCapacity** problems within a factor of $O(\log^* \Delta)$. Let us first formally define the problems and related terms.

A *schedule* for a set L of links is a partition of L into feasible subsets (or *slots*). The *length* of the schedule is its number of slots. The **Scheduling problem** is to find a minimum length schedule for a given set L . The length of an optimal schedule for L is denoted $OPTS(L)$.

The **WCapacity** problem is the generalized dual of **Scheduling**, where given a set L of links with weights $\omega : L \rightarrow \mathbb{R}^+$, the goal is to find a feasible subset $S \subseteq L$ of maximum weight $\sum_{i \in S} \omega(i)$.

Theorem 8. *There are polynomial $O(\log^* \Delta)$ -approximation algorithms for **Scheduling** and **WCapacity**. The approximation is obtained by coloring the graph $\mathcal{G}_{\gamma \log}$ (for an appropriate constant $\gamma \geq 1$) in the case of **Scheduling** and by approximating its maximum weighted independent set in the case of **WCapacity**.*

Proof. First consider the **Scheduling** problem. Let L be an input to **Scheduling**. We construct and color the graph $\mathcal{G}_{\gamma \log}(L)$ with constant γ chosen as in Corollary 1. By Corollary 1, such a coloring corresponds to a feasible schedule.

To derive the approximation factor, observe on one hand that in view of Thms. 5 and 6, any schedule of L can be refined into a coloring of $\mathcal{G}_\gamma(L)$ with only constant factor increase in the number of slots. Thus, $OPTS(L) = \Omega(\chi(\mathcal{G}_\gamma(L)))$. On the other hand, by Thm. 1, $\chi(\mathcal{G}_{\gamma \log}(L)) = O(\log^*(\Delta)) \cdot \chi(\mathcal{G}_\gamma(L)) = O(\log^* \Delta) \cdot OPTS(L)$. It is readily verified that the function $\gamma \log$ is strongly sub-linear, implying, via Thm. 3, that $\mathcal{G}_{\gamma \log}(L)$ is constant-simplicial and thus colorable within constant approximation factor.

Now consider the **WCapacity** problem. Let a set L be given. As in the case of **Scheduling**, we first construct the graph $\mathcal{G}_{\gamma \log}(L)$ with constant γ chosen as in Corollary 1. We find a constant-factor approximate weighted maximum independent set in $\mathcal{G}_{\gamma \log}(L)$ using the fact that this graph is constant-simplicial. By Corollary 1, the resulting set is feasible, i.e. it is a valid solution for **WCapacity**. Now let us derive the approximation factor. Let W_l and W_u be the weights of the weighted maximum independent sets in $\mathcal{G}_\gamma(L)$ and $\mathcal{G}_{\gamma \log}(L)$ respectively, and let W_o be the weight of the optimal solution to **WCapacity** in L . Let S be a solution to **WCapacity** in L . Since S is feasible, it can be split into a constant number of γ -independent subsets, by Thms. 5 and 6. Let S' be the largest weight subset. Obviously, the weight of S' is $\Omega(W_o)$, implying that $W_l = \Omega(W_o)$, as S' is an independent set in \mathcal{G}_γ . On the other hand, Thm. 1 implies that S' can be refined into at most $O(\log^* \Delta)$ $\gamma \log$ -independent subsets. The largest weight subset will have weight at least $\Omega(W_l / \log^* \Delta)$, which implies that $W_u = \Omega(W_l / \log^* \Delta) = \Omega(W_o / \log^* \Delta)$. \square

7.2 Measure of Interference

While approximation algorithms give bounds relative to an optimal value, it is frequently advantageous to have bounds in terms of some intrinsic parameters or more easily computable properties. Thus the interest in bounding chromatic numbers of graphs in terms of clique numbers, broadcast algorithms in terms of network diameter, and routing time in terms of “congestion + dilation”. Our results also imply bounds for the optimum schedule length that can be efficiently computed from the network topology. Previous such results involved logarithmic factors in n and/or Δ [9, 31].

Let G be a k -simplicial graph and let v_1, v_2, \dots, v_n be a k -simplicial elimination order of vertices, which for our conflict graphs is by increasing link length. A k -approximate coloring of G is obtained

by coloring the vertices greedily in reverse order. The number of colors used is at most the maximum post-degree plus 1, or $\max_i \{|N(v_i) \cap \{v_{i+1}, \dots, v_n\}|\} + 1 \leq k \cdot \chi(G) + 1$. We therefore define

$$B_f(L) = \max_{i \in L} |\{j \in L : l_j \geq l_i, d(i, j) \leq l_i f(l_j/l_i)\}|,$$

for a function f , and observe that $\chi(\mathcal{G}_f(L)) = \Theta(B_f(L))$. The results of Sec. 6 and 8 then imply the following theorem.

Theorem 9. *There are constants $a, b > 0$ and $\gamma \geq 1$, such that for any set L ,*

$$a \cdot B_\gamma(L) \leq \text{OPTS}(L) \leq b \cdot B_{\log}(L) \text{ and } \frac{B_{\log}(L)}{B_\gamma(L)} = O(\log^* \Delta(L)).$$

Moreover, there are infinitely many instances L' and L'' s.t. $\frac{\text{OPTS}(L')}{B_\gamma(L')} = \Omega(\log^* \Delta(L'))$ and $\frac{B_{\log}(L'')}{\text{OPTS}(L'')} = \Omega(\log^* \Delta(L''))$.

7.3 A Necessary and Sufficient Condition for Feasibility

Another interesting implication of Thm. 7 is the following result that shows that the sufficient condition for feasibility stated in Thm. 4 is essentially necessary in doubling metric spaces. This result is of independent interest, as it may prove useful for improved analysis of various problems. It should be noted that this theorem does not hold in general metric spaces.

The proof consists of two parts, bounding the influence on a link i by faraway links (i.e., links that are highly independent from link i) on one hand using Thm. 7, and by near links (the rest) on the other hand, using simple manipulations of the SINR condition.

Theorem 10. *Let L be a 3^α -feasible set of links. Then, $I(L) = O(1)$.*

Proof. Let us fix a link $i \in L$ and denote $S = L_i^-$. We split S into two subsets S_1 and S_2 , where for each link $j \in S_1$, j and i are f -independent with $f(x) = 2x$, and $S_2 = S \setminus S_1$.

Recall that by Thm. 6, 3^α -feasibility implies 2-independence of S_1 . The bound $I(S_1, i) = O(1)$ then follows by applying an analogue of Thm. 7 with $\gamma = 1$ and with f -independence instead of $\widehat{\log}$ -independence, which can be done because $\widehat{\log}(x) = O(f(x))$.

It remains to show that $I(S_2, i) = O(1)$. Let P be a power assignment for which L is P -feasible. Then, the SINR condition gives us the following inequalities:

$$\frac{P(i)}{l_i^\alpha} > 3^\alpha \sum_{j \in S_2} \frac{P(j)}{d_{ji}^\alpha}, \text{ and } \frac{P(j)}{l_j^\alpha} > 3^\alpha \frac{P(i)}{d_{ij}^\alpha} \text{ for all } j \in S_2.$$

By replacing $P(j)$ with $3^\alpha \frac{P(i)l_j^\alpha}{d_{ij}^\alpha}$ in the first inequality and simplifying the expression, we get:

$$\sum_{j \in S_2} \frac{l_i^\alpha l_j^\alpha}{d_{ij}^\alpha d_{ji}^\alpha} \leq 9^{-\alpha}. \quad (11)$$

In order to extract a bound on $I(S_2, i)$ from (11), we will show that one of the values $d_{ij}^\alpha, d_{ji}^\alpha$ in the denominator can be canceled out with l_i^α in the numerator and the other one can be replaced with $d(i, j)^\alpha$ by only introducing additional constant factors in the expression. Such a modification will transform the left side of (11) into $I(S_2, i)$.

Let us assume w.l.o.g. that $d_{ij} \geq d_{ji}$. Recall that for each $j \in S_2$, $d(i, j) \leq 2l_i$ by definition of S_2 . Using the triangle inequality, we obtain $d_{ij} \leq d(i, j) + l_i + l_j \leq 4l_i$. On the other hand, as it was mentioned above, the set S_2 is 2-independent, which implies that $d_{ji} \geq d(i, j) > 2l_j$. Using the triangle inequality again, we obtain:

$$d(s_i, s_j) \geq d_{ij} - l_j \geq d_{ji} - l_j > d_{ji}/2 \text{ and } d(r_i, r_j) \geq d_{ji} - l_j > d_{ji}/2.$$

Thus, $d(i, j) > d_{ji}/2$. By replacing d_{ij} with $4l_i$ and d_{ji} with $2d(i, j)$ in the left-hand part of (11), we obtain the desired bound: $I(S_2, i) \leq (8/9)^\alpha$. Since this holds for an arbitrary $i \in L$, we get that $I(L) = O(1)$. \square

Remark. Note that Thm. 5 implies that any feasible set can be refined into a constant number of 3^α -feasible subsets. Thus, the influence function fully captures feasibility in fading metrics, modulo constant factors.

8 Limitations of the Graph-Based Approach

We have found that conflict graphs can achieve a remarkably good, yet super-constant, approximation for scheduling problems in doubling metrics. We examine in this section how far this approach can be pushed, obtaining essentially tight bounds. We treat these issues in terms of the Scheduling problem.

In the first part of the section, we expose the limitations of the graph method in Euclidean spaces. We show, in particular, that conflict graphs do not yield any non-trivial approximation to the Scheduling problem in terms of the number of links n . In particular, they cannot lead to constant factor approximation. We also consider approximation limits in terms of the parameter Δ , and show that for all reasonable functions f , the approximation factor is at least $\Omega(\log^* \Delta)$. Thus, the approximation factor we obtained cannot be improved within a conflict-graph framework. Note that the instances we construct are embedded on the real line, i.e., in one dimensional space.

In the second part of the section, we find that the graph method cannot provide any non-trivial approximation guarantees in general metric spaces, neither in terms of n nor Δ .

8.1 Euclidean Spaces

In the following theorem, we construct, for any function $f = \omega(1)$, a feasible set of f -adjacent links. The construction is based on the following observations. On the one hand, it follows from Thm. 4 that any set of exponentially growing links arranged sequentially by the order of length on the real line is (almost) feasible. On the other hand, given such a set S of links on the line, a new link j can be formed so that j is f -adjacent to all the links in S while the set $S \cup j$ stays feasible; the only requirement is that j be long enough. Our construction then builds recursively on these ideas.

Theorem 11. *Let $f(x) = \omega(1)$. For any integer $n > 0$, there is a feasible set L of n links arranged on the real line, such that $\mathcal{G}_f(L)$ is a clique, i.e., $\chi(\mathcal{G}_f(L)) = n$. Moreover, if $f(x) \geq g(x)$ ($x \geq 1$) for a strongly sub-linear increasing function $g(x)$ with $g(x) = \omega(1)$, then $n = \Omega(g^*(\Delta))$.*

Proof. Consider a set of cn links $\{1, 2, \dots, cn\}$ arranged sequentially from left to right on the real line, where $c > 0$ is a constant to be chosen later. Each link i is directed from left to right and for each $i = 1, 2, 3, \dots, n-1$, the nodes s_{i+1} and r_i share the same location on the line, i.e.,

$r_i = s_i + l_i = s_{i+1}$. See Figure 1. The lengths of links are defined inductively, as follows. We set $l_1 = 1$, and for $i \geq 1$, we choose l_{i+1} to be the minimum value satisfying:

$$l_{i+1} \geq 2l_i \quad (12)$$

$$2d(i+1, j) = 2d_{i+1, j} \leq l_j f(l_{i+1}/l_j) \text{ for all } j \leq i. \quad (13)$$

Such a value of l_{i+1} can be chosen as follows. By the inductive hypothesis, we have $l_j \geq 2l_{j-1}$ for $j = 2, 3, \dots, i$. This implies that $l_i \geq \sum_{j=1}^{i-1} l_j$. Then, we have that $d_{i+1, j} = \sum_{t=j+1}^i l_t \leq 2l_i$ for $j = 1, 2, \dots, i$. Thus, it is enough to choose l_{i+1} so that $l_{i+1} \geq 2l_i$ and $4l_i \leq l_j f(l_{i+1}/l_j)$, which can be done using $f = \omega(1)$ and the fact that the values of l_j for $j = 1, 2, \dots, i$ are already fixed at this point. This completes the construction. First note that (13) implies that $\mathcal{G}_f(L)$ is a clique. It

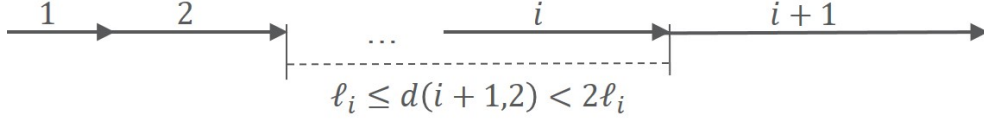


Figure 1: The construction in Thm. 11.

remains to argue feasibility. Consider the odd numbered links $S = \{1, 3, \dots, 2t+1\}$. Let us fix a link $2k+1 \in S$. Note that for each $j \in S_{2k+1}^-$, $d(j, 2k+1) \geq l_{2k}$. We have that

$$I(S_{2k+1}^-, 2k+1) = \sum_{j \in S_{2k+1}^-} \frac{l_j^\alpha}{d(j, 2k+1)^\alpha} \leq \sum_{j \in S_{2k+1}^-} \left(\frac{l_j}{l_{2k}} \right)^\alpha \leq \sum_{j \in S_{2k+1}^-} \frac{l_j}{l_{2k}} \leq 1,$$

where the second inequality holds because $l_j/l_{2k} \leq 1$ and the last inequality follows from (12). Thus, we can extract a constant fraction S' of S that is feasible, using Thm. 5. With the right choice of the constant c in the beginning of the proof we have that $|S'| = n$. This proves the first part of the theorem.

Now let us assume that $f(x) \geq g(x)$ for a strongly sub-linear function $g(x)$ with $g(x) = \omega(1)$. Then, there is a constant x_0 such that $g(x) < x$ for all $x \geq x_0$ (because $g(x) = o(x)$) and there is a constant c such that $2g(x)/x \leq g(y)/y$ whenever $x \geq cy$ (strong sub-linearity). In this case we repeat the construction above with slight modifications.

We set $l_1 = 1$ and set $l_{i+1} > \max\{c, x_0\}$ be the minimum value s.t. $g(l_{i+1}) \geq 2l_i$, for $i = 1, 2, \dots$ (such a value exists because $g(x) = \omega(1)$). Let us show that the conditions (12-13) hold with these lengths.

Since $l_{i+1} \geq x_0$, we have that $l_{i+1} > g(l_{i+1}) \geq 2l_i$, which implies (12). This in turn implies, as observed in the first part of the proof, that $d(i+1, j) < 2l_i$ for all $2 \leq j \leq i$. Let us denote $x = l_{i+1}/l_1 = l_{i+1}$ and $y = l_{i+1}/l_j$. Note that $x/y = l_j \geq c$, so we have, by strong sub-linearity of g , that $g(y)/y \geq 2g(x)/x$, or equivalently, that $l_j \cdot g(l_{i+1}/l_j) \geq 2 \cdot g(l_{i+1})$; hence $l_j \cdot g(l_{i+1}/l_j) \geq 4l_i > 2d(i+1, j)$ for all $2 \leq j \leq i$, which means that (13) also holds.

It remains to prove the lower bound for n . Recall that the value of l_{i+1} is the minimum satisfying $g(l_{i+1}) \geq 2l_i$ for $i = 1, 2, \dots, n-1$. Then, we have $g(l_{i+1}/2) < 2l_i$ or, equivalently, $h(l_{i+1}/2) < l_i/2$, where $h(x) = g(x)/4$. Thus,

$$1/2 = l_1/2 > h(l_2/2) > h(h(l_3/2)) > \dots > h^{(n-1)}(l_n/2) = h^{(n-1)}(\Delta/2),$$

which implies that $n = \Omega(h^*(\Delta/2)) = \Omega(g^*(\Delta))$. \square

Corollary 2. *In terms of the number of links n , the approximation factor for Scheduling when using \mathcal{G}_f with any $f = \omega(1)$ is no better than n .*

By choosing $g(x) = \gamma \widehat{\log}(x)$ in Thm. 11, we obtain that the approximation factor of $O(\log^* \Delta)$ cannot be improved for $\mathcal{G}_{\gamma \widehat{\log}}$.

Corollary 3. *Let $f(x) = \Omega(\log^{(c)} x)$ for a constant c . Then, for each $\Delta > 0$, there is a feasible set of links L with $\Delta(L) = \Omega(\Delta)$, such that $\mathcal{G}_f(L)$ is a clique of size $\Theta(\log^* \Delta(L))$.*

While the theorem above shows that graphs \mathcal{G}_f with $f = \Omega(\log^{(c)} x)$ for some constant c require too much separation, the theorem below shows that graphs \mathcal{G}_f with $f = O(\log^{1/\alpha} x)$ provide insufficient separation, leading, perhaps surprisingly, to a similar sized gap of $\log^* \Delta$. Namely, $\chi(\mathcal{G}_f(L')) = \frac{OPTS(L')}{\Omega(\log^* \Delta)}$ holds for certain instances L' . The construction follows the general structure of Thm. 7 in [20] of a lower bound for scheduling the edges of a minimum spanning tree of a set of points in the plane. There are two technical challenges to overcome, in order to implement this structure in our setting. First, the construction of [20] is not f -independent. Second, even when ignoring the f -independence requirement, the lower bound for the scheduling number obtained in [20] is only $\Omega(\log \log^* \Delta)$.

Theorem 12. *Let $f(x) = O(\log^{1/\alpha} x)$. For each $\Delta > 0$, there is an f -independent set of links L on the real line with $\Delta(L) = \Omega(\Delta)$ that cannot be scheduled in fewer than $\Theta(\log^* \Delta(L))$ slots.*

We describe the idea of the construction informally. The construction is inductive, starting from a trivial instance L_1 containing a single link. For $t \geq 1$, assume there is an instance L_t having the desired properties, i.e., L_t is f -independent and with $OPTS(L) \geq t$. In order to construct the instance L_{t+1} , consider a single link j that is longer than the links in L_t and place it at distance d from L_t so that all the links in L_t are f -independent from j . Let I_0 denote the minimum influence of a link from L_t on link j . Now, take k identical copies of L_t and place them at a distance d from j . This will of course violate the independence between different instances, which we will address shortly, but they will still be independent from link j . The idea is that if the number of copies k is large enough, then for any set S containing at least one link from each copy, we will have $I(S, j) = kI_0 > c_0$, where c_0 is a constant large enough to ensure that $S \cup \{j\}$ is infeasible (based on Thm. 10). This will mean that any schedule of the link j and the k copies must place at least *one whole copy* of L_t in slots separate from j . Since it takes at least t slots to schedule one copy of L_t , it takes at least $t + 1$ slots to schedule all the copies together with link j .

It remains to address the issue of f -independence between different copies. Note that because of the scale-invariance of the influence operator, we can scale a copy of L_t by a factor s and place it further than before, at a distance $s \cdot d$ from link j and still have the minimum influence of I_0 on j . However, in order for this influence to be taken into the account, the link j must still be longer than the links in the scaled instance. Using this trick, we can scale different copies by different factors and guarantee their mutual independence, while preserving the properties we had in the case of identical copies. Since the link lengths must grow exponentially at each step t , the number t of slots required will be small compared to the number of links and the parameter Δ , but will still be $\Omega(\log^* \Delta)$.

Proof. For a set S of links, we will use $\text{diam}(S)$ to denote the diameter of S , or the maximum distance between nodes in S .

We will construct a set of links that cannot be scheduled in fewer than $\Theta(\log^* \Delta)$ 3^α -feasible slots, relying on the necessary condition for feasibility (Thm. 10). This will be sufficient to prove

the theorem, as Thm. 5 will imply that there cannot exist a β -feasible schedule with $\Theta(\log^* \Delta)$ slots for that set, for any constant β .

Let us fix a function f . Note that since $f = O(\log^{1/\alpha})$, there is a constant $C \geq 1$ s.t. $f(x) \leq C \log^{1/\alpha} x$. We construct sets L_t of links recursively. The construction is illustrated in Figure 2. All the links will be arranged on the real line and the receiver of each link will be to the right of the sender. Initially, we have a set L_1 consisting of a single link of length 1, for which a single slot is sufficient and necessary. Suppose that we have already constructed L_t with the property that at least t slots are required for scheduling L_t . The instance L_{t+1} is constructed as follows using k scaled copies of L_t , where k is to be determined. First we place a single very long link j_{t+1} in the line. We then add, in order from left to right, copies $L_t^1, L_t^2, \dots, L_t^k$ of L_t to the right of j_{t+1} , where L_t^s is the copy of L_t scaled by a factor 8^s . The idea is to make the construction so that the following properties hold:

- (i) L_{t+1} is f -independent,
- (ii) $t = \Omega(\log^* \Delta(L_t))$,
- (iii) for any set $S = \{i_1, i_2, \dots, i_k\}$ with $i_s \in L_t^s$, $s = 1, 2, \dots, k$, we have that $I(S, j_{t+1}) > c_0$ for a constant c_0 of our choice.

The last property ensures that each 3^α -feasible schedule of L_{t+1} must put a whole copy L_t^s in a slot separate from j_{t+1} . Indeed, if there was a schedule that placed at least one link from each copy L_t^s in the same slot with j_{t+1} then we would get a contradiction with (iii): we would have $I(S, j_{t+1}) = O(1)$ for some S as above, due to Thm. 10. Recall that L_t needs at least t slots to be scheduled, and so does each copy of it. It follows that L_{t+1} needs at least $t + 1 = \Omega(\log^* \Delta(L_t))$ slots to be scheduled, one for j_t and at least t for scheduling the copies of L_t . Proving the properties (i-iii) will complete the proof of the theorem.

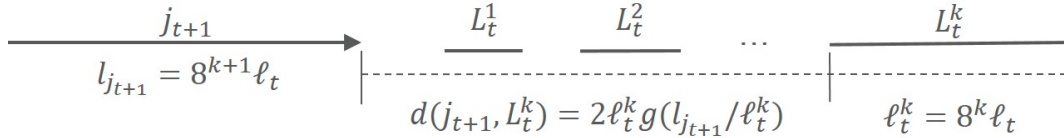


Figure 2: The recursive construction of L_{t+1} .

Now let us describe the inductive step of the construction in detail. Let $\ell_t = \text{diam}(L_t)$ denote the diameter of L_t . The number of copies of L_t is $k = 2^{c\ell_t}$ for a large enough constant c . The length of link j_{t+1} is set to $l_{j_{t+1}} = 8^{k+1}\ell_t$. It remains to specify the placement of each copy L_t^s so as to guarantee the desired properties of L_{t+1} .

We assume by induction that the links within each copy of L_t are f -independent. We place the copies L_t^s so that the links between any two copies are f -independent and are f -independent from j_{t+1} . Let $\ell_t^s = \text{diam}(L_t^s) = 8^s \ell_t$ denote the diameter of L_t^s . Let $g(x) = C \log^{1/\alpha} x$. We place each copy L_t^s at a distance $d(L_t^s, j_{t+1}) = 2\ell_t^s g(l_{j_{t+1}}/\ell_t^s)$ from j_{t+1} . The construction is ready.

We first prove the property (i).

Claim 4. *With the distances defined as above, the set L_{t+1} is f -independent.*

Proof. Consider any link $i \in L_t^s$. We have that

$$d(i, j_{t+1}) \geq d(L_t^s, j_{t+1}) = 2\ell_t^s g(l_{j_{t+1}}/\ell_t^s) \geq 2l_i g(l_{j_{t+1}}/l_i) \geq 2l_i f(l_{j_{t+1}}/l_i),$$

where the second inequality follows from the fact that $xg(c/x)$ is an increasing function of x and that $l_i < \ell_t^s$, and the third inequality follows because $f(x) \leq g(x)$ for all x . Thus, all the links in L_t^s are f -independent from j_{t+1} . Now let us show that any two links i, k with $l_i \leq l_k$ from different copies L_t^s and L_t^r with $s > r$ are f -independent (no matter which link is from which copy). Since $f(x) \leq g(x)$, it will be enough to show that

$$d(i, k) > l_i g(l_k/l_i). \quad (14)$$

Recall that $xg(c/x)$ is an increasing function of x . Then, for a fixed k , the right side of (14) is maximized when l_i is maximum. On the other hand, for a fixed i , the value $g(l_k/l_i)$ is maximized when l_k is maximum, because g is an increasing function. Let j_t denote the maximum length link in L_t . Then, the maximum link length in L_t^s (in L_t^r) is $8^s l_{j_t}$ ($8^r l_{j_t}$). Therefore, it is enough to show that

$$d(i, k) > \ell_t^r g(8^s l_{j_t} / (8^r l_{j_t})) = \ell_t^r g(8^{s-r}) = C(3(s-r))^{1/\alpha} \ell_t^r.$$

We have that

$$d(i, k) \geq d(L_t^s, L_t^r) = d(L_t^s, j_{t+1}) - d(L_t^r, j_{t+1}) - \ell_t^r \geq 2\ell_t^s g(l_{j_{t+1}}/\ell_t^s) - 3\ell_t^r g(l_{j_{t+1}}/\ell_t^r).$$

The term $g(l_{j_{t+1}}/\ell_t^r)$ can be bounded by

$$g(l_{j_{t+1}}/\ell_t^r) = g(8^{s-r} l_{j_{t+1}}/\ell_t^s) \leq 3^{s-r} g(l_{j_{t+1}}/\ell_t^s),$$

where the last inequality follows because $g(8x) \leq 3g(x)$ for $x \geq 2$ (note that $\alpha \geq 1$). Thus,

$$d(i, k) \geq 2\ell_t^s g(l_{j_{t+1}}/\ell_t^s) - 3^{s-r+1} \ell_t^r g(l_{j_{t+1}}/\ell_t^s) > C(2 \cdot 8^{s-r} - 3 \cdot 3^{s-r}) \ell_t^r > C(3(s-r))^{1/\alpha} \ell_t^r.$$

□

Next, we can observe that (the first line follows because the links are arranged linearly)

$$\begin{aligned} \ell_{t+1} &= l_{j_{t+1}} + d(L_t^k, j_{t+1}) + \ell_t^k \\ &\leq l_{j_{t+1}} + 2\ell_t^k g(l_{j_{t+1}}/\ell_t^k) + \ell_t^k \\ &= 8^{k+1} \ell_t + 8^k \ell_t g(8) + 8^k \ell_t \\ &= O(8^{2^{c\ell_t}}). \end{aligned} \quad (15)$$

Since the minimum link-length in L_{t+1} is 1, we can conclude that $\Delta(L_t) < \ell_t \leq 2 \uparrow (c_1 t)$ for a constant c_1 and for each t , where \uparrow denotes the tower function. This implies that $t = \Omega(\log^* \Delta(L_t))$. The property (ii) is now proven.

It remains to check that (iii) holds. Let us consider a link i_s from L_t^s where i_s is the copy of link i in L_t . We have that

$$d(i_s, j_t) \leq \ell_t^s + d(L_t^s, j_t) = \ell_t^s + 2C\ell_t^s \log^{1/\alpha} (l_{j_{t+1}}/\ell_t^s) \leq c_2 \ell_t^s (k-s+1)^{1/\alpha},$$

for a constant c_2 . This implies:

$$I(i_s, j_{t+1}) = \left(\frac{l_{i_s}}{d(i_s, j_{t+1})} \right)^\alpha \geq \left(\frac{l_{i_s}}{c_2 (k-s+1)^{1/\alpha} \ell_t^s} \right)^\alpha \geq \frac{1}{c_3 (k-s) \ell_{t-1}},$$

where we used the fact that $l_{i_s}/\ell_t^s = l_i/\ell_t \geq 1/\ell_t$. Now, let i_s , $s = 1, 2, \dots, k$ be a set of links where $i_s \in L_t^s$ and they are not necessarily the copies of the same link of L_t . Then,

$$I(S, j_{t+1}) = \sum_{s=1}^k I(i_s, j_{t+1}) > \sum_{s=1}^k \frac{1}{c_3 (k-s+1) \ell_t} = \Omega\left(\frac{\log k}{\ell_t}\right).$$

Recall that $k = 2^{c\ell_t}$. By taking the constant c large enough, we can thus guarantee the property (iii). This completes the proof of all the properties of L_t and the proof of the theorem. □

8.2 General Metric Spaces

The following theorem shows that conflict graphs can be arbitrarily far from schedules in general metric spaces. Given a function f , the construction consists of an f -independent set of *unit length* links. Since all links have length 1, f -independence is equivalent to $f(1)$ -independence. The separation between the links is just enough to ensure $f(1)$ -independence. However, since all the links are equally ($f(1)$ -) separated from any given link, their interference accumulates and only a constant number of links can be scheduled in the same slot. This leads to schedules of length $\Theta(n)$.

Proposition 3. *For each function f and any $n \geq 1$, there is an f -independent set of n unit length links (hence, $\Delta = 1$) that cannot be scheduled into less than $\Theta(n)$ slots.*

Proof. Let $L = \{1, 2, \dots, n\}$ be the set of links. We define the lengths and the distances between the links such as to ensure the metric constraints hold. For each link i we define $l_i = 1$. The distances between the nodes are defined as follows:

- ✓ sender to sender: $d(s_i, s_j) = f(1) \cdot (l_i + l_j) = 2f(1)$,
- ✓ sender to receiver: $d(s_i, r_j) = d(s_i, s_j) + l_j = 2f(1) + 1$,
- ✓ receiver to receiver: $d(r_i, r_j) = d(s_i, s_j) + l_i + l_j = 2f(1) + 2$.

It is straightforward to check that such distances define a metric. Moreover, the whole set of links in this metric is f -independent, since $d(i, j) > f(1) \cdot l_i = l_i f(l_j/l_i)$. Let us consider any P -feasible subset S of k links for a power assignment P . Let us fix a link $i \in S$. The SINR condition implies: $P(i) > \beta \sum_{j \in S \setminus \{i\}} \frac{P(j)l_i^\alpha}{d_{ji}^\alpha}$ and $P(j) > \beta \frac{P(i)l_j^\alpha}{d_{ij}^\alpha}$ for all $j \in S \setminus \{i\}$. By replacing $P(j)$ with $\frac{\beta P(i)l_j^\alpha}{d_{ij}^\alpha}$ in the first inequality and canceling the term $P(i)$, we obtain:

$$1 > \beta^2 \sum_{j \in S \setminus \{i\}} \frac{l_i^\alpha l_j^\alpha}{d_{ij}^\alpha d_{ji}^\alpha} = \beta^2 \sum_{j \in S \setminus \{i\}} \frac{1}{(2f(1) + 1)^2} = \frac{\beta^2(|S| - 1)}{(2f(1) + 1)^2},$$

which implies that $|S| < \left(\frac{2f(1)+1}{\beta}\right)^2 + 1 = O(1)$. Since S was an arbitrary feasible subset of L , we conclude that L cannot be split into less than $\Theta(n)$ feasible subsets. \square

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